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Pastings of MV-Effect Algebras

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We give a method of constructing a lattice effect algebra E with given family of blocks. The given MV-effect algebras are "pasted" along some common sub-MV-effect algebras in a such manner that there exists an ((o)-continuous) state on the pasting E.

KEY WORDS: lattice effect algebra; MV-effect algebra; block; state; isotropic index.

1. INTRODUCTION

Lattice effect algebras generalize orthomodular lattices and MV-effect algebras (including Boolean algebras), as well as their common horizontal sums and direct products. For instance, the horizontal sum of two MV-effect algebras, or, an orthomodular lattice and an MV-effect algebra is a lattice effect algebra. The same holds for their direct product.

Effect algebras have been introduced by Foulis and Bennet (1994). In some sense equivalent structures D-posets have been introduced by Kôpka and Chovanec (1994). Elements of these structures represent events which may be unsharp or imprecise, e.g., quantum effects or fuzzy events.

In Riečanová (2000b) it has been shown that every lattice effect algebra E is a union of its maximal subsets of pairwise compatible elements, called blocks. Blocks are sub-effect algebras and sublattices of E. Moreover, every block M of E is an MV-effect algebra in its own right, which means that it can be organized into an MV-algebra (Chang, 1958). It follows that the intersection of two or more blocks of E is again an MV-effect algebra which is a sub-effect algebra and a sublattice of E. Unfortunately, in spite of the fact that there is a state on every MV-effect algebra (MV-algebra), there are even finite lattice effect algebras admitting no states (Greechie, 1971).

The aim of this paper is to present a method of constructing a complete (atomic) effect algebra E with given family of blocks in a such manner that a state

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(an (o)-continuous state in the case of atomic E) will exist on E. One well-known kind of such "pasting" of blocks is their horizontal sum, which is the disjoint union of blocks with "pasted" (identified) all least and all greatest elements. Our aim is to "past" MV-effect algebras together along some isomorphic sub-MV-effect algebras of given blocks.

2. BASIC DEFINITIONS AND FACTS

Definition 2.1. (Foulis and Bennett (1994)) A partial algebra $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. Moreover, if we write $a \oplus b = c$ for $a, b, c \in E$, then we mean both that $a \oplus b$ is defined and $a \oplus b = c$. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the operation $\ominus$ and the relation $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). Moreover, if E is a modular or distributive lattice then E is called *modular* or *distributive* effect algebra.

Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if

(i) $1 \in Q$,
(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

Assume that $(E_1; \oplus_1, 0_1, 1_1)$ and $(E_2; \oplus_2, 0_2, 1_2)$ are effect algebras. An injection $\varphi : E_1 \to E_2$ is called an *embedding* if $\varphi(1_1) = 1_2$ and for $a, b \in E_1$ we have $a \leq b'$ iff $\varphi(a) \leq (\varphi(b))'$ in which case $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$. We can easily see that then $\varphi(E_1)$ is a sub-effect algebra of E_2 and we say that E_1 and $\varphi(E_1)$ are *isomorphic* (written $E_1 \cong \varphi(E_1)$) , or that E_1 *is up to isomorphism a sub-effect algebra of* E_2. We usually identify E_1 with $\varphi(E_1)$.

For an effect algebra E a map $\omega\ E \to [0, 1] \subseteq (-\infty, \infty)$ is called a *state* on E if $\omega(1) = 1$ and $x \leq y'$ implies $\omega(x \oplus y) = \omega(x) + \omega(y)$; ω is called (o)-continuous if $x_\alpha \overset{(o)}{\to} x$ implies $\omega(x_\alpha) \to \omega(x)$. Here, for a net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of E, we write $x_\alpha \overset{(o)}{\to} x$ if there exists a nondecreasing net $(u_\alpha)_{\alpha\in\mathcal{E}}$ and a nonincreasing net $(v_\alpha)_{\alpha\in\mathcal{E}}$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$, for $\alpha \in \mathcal{E}$, and $u_\alpha \uparrow x$, $v_\alpha \downarrow x$, which means that $\bigvee_{\alpha\in\mathcal{E}} u_\alpha = x = \bigwedge_{\alpha\in\mathcal{E}} v_\alpha$. It is easy to show that ω is (o)-continuous iff $x_\alpha \uparrow x$ implies $\omega(x_\alpha) \uparrow \omega(x)$.

An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$ the elements $ne = \underbrace{e \oplus e \cdots \oplus e}_{n\text{-times}}$ exist for all $n \in N$. An Archimedean effect algebra is called *separable* if every $\oplus$-orthogonal system of elements of E is at most countable. We can show that *every complete effect algebra is Archimedean* (Riečanová, 2000a).

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if nx exists for every $n \in N$. We write $\mathrm{ord}(x) = n_x \in N$ if n_x is the greatest positive integer such that $n_x x$ exists in E. Then n_x is called an *isotropic index* of x. Clearly, in an Archimedean effect algebra $n_x < \infty$ for every $x \in E$.

For more details we refer the reader to Dvurečenskij and Pulmannová (2000).

3. THE PASTING OF A FAMILY OF MV-EFFECT ALGEBRAS

From now on we make the assumption that E is a lattice effect algebra.

Recall that elements x and y of a lattice effect algebra E are called *compatible* (written $x \leftrightarrow y$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$. For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements of E are compatible then E is called an *MV-effect algebra*. Further, every maximal subset M of pairwise compatible elements of a lattice effect algebra E is a sub-effect algebra and a sublattice of E, called a *block* of E. Moreover, every block is an MV-effect algebra in its own right and E is a union of its blocks (see Riečanová, 2000b). Thus E is a "pasting" of MV-effect algebras. Every MV-effect algebra M has the *Riesz decomposition property* (RDP, for short): $(c, a, b \in M$ with $c \leq a \oplus b) \Longrightarrow$ (there is $a_1 \leq a$ and $b_1 \leq b$ such that $c = a_1 \oplus b_1$).

Recall that a direct product $\prod\{E_\kappa \mid \kappa \in H\}$ of effect algebras E_κ is a Cartesian product with $\oplus$, 0 and 1 defined "coordinatewise." An element $z \in E$ is called *central* if the intervals $[0, z]$ and $[0, z']$ with the inherited $\oplus$-operation are effect algebras in their own right and $E \cong [0, z] \times [0, z']$, (see Greechie *et al.*, 1995). The set $C(E) = \{z \in E \mid z \text{ is central}\}$ is called a *center* of E. If $C(E) = \{0, 1\}$ then E is called *irreducible*.

In every lattice effect algebra E the set $S(E) = \{x \in E \mid x \wedge x' = 0\}$ is an orthomodular lattice and $B(E) = \{x \in E \mid x \leftrightarrow E\}$ is an MV-effect algebra such that $S(E)$ and $B(E)$ are sub-lattices and sub-effect algebras of E. Moreover, $z \in C(E)$ iff $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$, which gives $C(E) = B(E) \cap S(E)$

for every lattice effect algebra E. If E is an MV-algebra then $C(E) = S(E)$. Further, $S(E)$, $B(E)$ and $C(E)$ are closed with respect to all existing infima and suprema (Riečanová, 2001). In general, $C(E) = \{0, 1\}$ does not imply $C(S(E)) = \{0, 1\}$. Note that $S(E)$ is called the set of *sharp elements* of E and $B(E)$ is called the *compatibility center of* E.

Recall that the *length* of a finite chain is the number of its elements minus 1. The *length* (*height*) of a lattice L is finite if the supremum over the number of elements of chains in L equals to some natural number n and then $n - 1$ is called *length of the lattice* L.

Every *finite chain* $0 < a < 2a < \cdots < 1 = n_a a$ is a distributive effect algebra in which every pair of elements is compatible, hence it is an MV-effect algebra.

An element a of an effect algebra E is called an *atom* if $0 \le b < a$ implies $b = 0$ and E is called *atomic* if for every $x \in E$, $x \neq 0$ there is an atom $a \in E$ with $a \le x$. Clearly every finite effect algebra is atomic.

Definition 3.2. Let E be an effect algebra and let $(E_\chi)_{\chi \in H}$ be a family of sub-effect algebras of E such that:

(i) $E = \bigcup_{\chi \in H} E_\chi$.

(ii) If $x \in E_{\chi_1} \setminus \{0, 1\}$, $y \in E_{\chi_2} \setminus \{0, 1\}$ and $\chi_1 \neq \chi_2$, $\chi_1, \chi_2 \in H$, then $x \wedge y = 0$ and $x \vee y = 1$.

Then E is called a *horizontal sum* of effect algebras $(E_\chi)_{\chi \in H}$.

Example 3.3. If MV-effect algebras M_1 and M_2 are finite chains of different lengths then the horizontal sum of M_1 and M_2 is the unique lattice effect algebra E such that $\{M_1, M_2\}$ is the family of all blocks of E.

Really, assume that $M_1 = \{0, a, \ldots, n_a a\}$, $M_2 = \{0, b, \ldots, n_b b\}$ and $n_a < n_b$. Contrary to our claim, assume that $ka = \ell b$ for some $\ell \neq n_b$. Then $a \le ka = \ell b < b'$, which gives $a \leftrightarrow b$ and hence, by Riečanová (2000b), $M_1 \cup M_2$ is the set of pairwise compatible elements, a contradiction. Thus $ka = \ell b$ implies $\ell = n_b$ which gives $k = n_a$, because $n_b b \in S(E)$ while for $k < n_a$ we have $ka \notin S(E)$. This proves that $E = M_1 \cup M_2$ is the horizontal sum of its blocks M_1 and M_2.

If M_1 and M_2 have the same length, i.e., $n_a = n_b$ then M_1 and M_2 are isomorphic MV-effect algebras and we can identify them. In this case we will call a and b isotropically equivalent.

Definition 3.4. Let M_1 and M_2 be complete atomic MV-effect algebras, let A_1 and A_2 be the sets of all atoms of M_1 and M_2, respectively, and let $D_1 \subseteq A_1$ and $D_2 \subseteq A_2$. The sets D_1 and D_2 are called *isotropically equivalent* (written $D_1 \overset{\text{istr}}{\sim} D_2$) if there is a bijection $\varphi\, D_1 \to D_2$ such that $n_p = \operatorname{ord}(p) = n_{\varphi(p)} = \operatorname{ord}(\varphi(p))$

for all $p \in D_1$. If $D_1 = \{p\}$ and $D_2 = \{q\}$ then p and q are called *isotropically equivalent atoms.*

Example 3.5. Assume that M_1 and M_2 are complete atomic MV-effect algebras and $E = M_1 \cup M_2$ is an effect algebra such that $\{M_1, M_2\}$ is the family of all blocks in E. Then every atom p of E is an atom of M_1 or M_2 and conversely, since $p \leftrightarrow x$ iff $p \leq x$ or $p \leq x'$. Further, if $p \in M_1 \cap M_2$ then $\{0, p, 2p, \ldots, n_p p\} \subseteq M_1 \cap M_2$, because $x \leftrightarrow p$ gives $x \leftrightarrow kp$ for all kp existing in E, and hence the isotropic indices of p in M_1 and M_2 must coincide. Moreover, if for atoms $p \neq q$ we have $n_p p = n_q q$ then $p \not\leftrightarrow q$ and the interval $[0, n_p p]_E$ in E is the horizontal sum of chains $\{0, p, \ldots, n_p p\}$ and $\{0, q, \ldots, n_q q\}$. Otherwise we have $p \leftrightarrow q$, which implies $q \leq p \oplus q = p \vee q = n_p p$ and by RDP we obtain $q = p$, a contradiction. Moreover, $k_p = \ell p$ for $k < n_p$ implies that $q \leq kp \leq p'$, which again gives $p \leftrightarrow q$, a contradiction.

Theorem 3.1. *Let $(M_\chi)_{\chi \in H}$ be a family of complete atomic MV-effect algebras such that there are nonempty sets D_χ of atoms of M_χ, $\chi \in H$ satisfying $D_{\chi_1} \overset{\text{istr}}{\sim} D_{\chi_2}$, for every pair $\chi_1, \chi_2 \in H$. Let $u_\chi = \oplus\{n_p p \mid p \in D_\chi\} \neq 1_\chi$, and $[0_\chi, u'_\chi]_{M_\chi} \neq \{0_\chi, u'_\chi\}$ $\chi \in H$. Then for chosen $\chi_0 \in H$ and every $\chi \in H$:*

(i) $u_\chi \in C(M_\chi)$.
(ii) $[0_\chi, u_\chi]_{M_\chi} \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}}$.
(iii) *$F_\chi = [0_\chi, u_\chi]_{M_\chi} \cup [u'_\chi, 1_\chi]_{M_\chi} \cong F_{\chi_0}$ and F_{χ_0} is an MV-effect algebra.*
(iv) *There is a complete atomic effect algebra $E = \cup_{\chi \in H} M_\chi$, whose family of all blocks coincides with $(M_\chi)_{\chi \in H}$, $\cap_{\chi \in H} M_\chi = F_{\chi_0}$ and $E \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times G$, where G is the horizontal sum of all $[0_\chi, u'_\chi]_{M_\chi}$, $\chi \in H$.*
(v) *$M_{\chi_1} \cap M_{\chi_2} = F_{\chi_0}$, for any pair of blocks of E.*
(vi) *There is an (o)-continuous state on E.*

Proof:

(i) Let A_χ be the set of all atoms of M_χ, $\chi \in H$. By Riečanová (2002), Theorem 3.3, for every $x \in M_\chi$ there is a set $\{a_\alpha \in A_\chi \mid \alpha \in \mathcal{E}\}$ and positive integers k_α, $\alpha \in \mathcal{E}$ such that $x = \oplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$. Moreover, $x \in S(M_\chi)$ iff $k_\alpha = n_{a_\alpha} = \text{ord}\,(a_\alpha)$ for all $\alpha \in \mathcal{E}$. Since M_χ is an MV-effect algebra, we have $S(M_\chi) = C(M_\chi)$, which proves that $u_\chi \in C(M_\chi)$.
(ii) Since M_χ is an MV-effect algebra, it satisfies Riesz decomposition property. By part (i) of the proof and RDP we have that for every $p \in D_\chi$ the element $n_p p$ is an atom of $S(M_\chi) = C(M_\chi)$ and hence the interval $[0_\chi, n_p p]_{M_\chi}$ in M_χ is a finite chain $0_\chi < p < 2p < \cdots < n_p p$ (see

Riečanová, 2003b). Let $\varphi\ D_\chi \to D_{\chi_0}$ be a bijection satisfying $n_p = \mathrm{ord}\,(p) = \mathrm{ord}\,(\varphi(p)) = n_{\varphi(p)}$ for all $p \in D_\chi$. We can extend the mapping φ onto $[0_\chi, n_p p]_{M_\chi}$ by putting $\varphi(kp) = k\varphi(p)$ for all $k \le n_p$. Obviously, $\varphi\ [0_\chi, n_p p]_{M_\chi} \to [0_{\chi_0}, \varphi(n_p p)]_{M_{\chi_0}}$ is an isomorphism. Further $\{n_p p \mid p \in D_\chi\}$ is the set of all atoms of the center $C([0_\chi, u_\chi]_{M_\chi})$ which, by Riečanová (2003a), Lemma 4.3, gives $[0_\chi, u_\chi]_{M_\chi} \cong \prod\{[0_\chi, n_p p]_{M_\chi} \mid p \in D_\chi\} \cong \prod\{[0_{\chi_0}, \varphi(n_p p)]_{M_{\chi_0}} \mid p \in D_{\chi_0}\} \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}}$.

(iii) Using (ii) we obtain $F_\chi \cong [0_\chi, u_\chi]_{M_\chi} \times \{0_\chi, u'_\chi\} \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times \{0_{\chi_0}, u'_{\chi_0}\} \cong F_{\chi_0}$. Since F_{χ_0} is a sub-effect algebra and a complete sub-lattice of M_{χ_0}, we obtain that F_{χ_0} is a complete atomic MV-effect algebra as well.

(iv) Since the intervals $[0_\chi, u'_\chi]_{M_\chi}$ are complete sub-lattices of M_χ, these intervals, with $\oplus$ inherited from M_χ, as well as their horizontal sum G are complete atomic effect algebras. Moreover, $([0_\chi, u'_\chi]_{M_\chi})_{\chi \in H}$ is a family of blocks of G. Let us construct an effect algebra $E = \cup_{\chi \in H} M_\chi \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times G$ by such a way that we identify all $F_\chi = [0_\chi, u_\chi]_{M_\chi} \cup [u'_\chi, 1_\chi]_{M_\chi}$, $\chi \in H$, with the MV-effect algebra F_{χ_0} and, moreover, we make a horizontal sum of all $[0_\chi, u'_\chi]_{M_\chi}$ identifying all 0_χ with 0_{χ_0} and u'_χ with u'_{χ_0}. By Riečanová (2003c), every block of E is isomorphic to a direct product of $[0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}}$ and a block of G, and conversely, since $[0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}}$ is an MV-effect algebra. This proves that M is a block of E iff there is $\chi \in H$ such that $M \cong [0_\chi, u_\chi]_{M_\chi} \times [0_\chi, u'_\chi]_{M_\chi}$ and hence $M = M_\chi$.

(v) Let $\chi_1, \chi_2 \in H$. Then $M_{\chi_1} \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times [0_{\chi_1}, u'_{\chi_1}]_{M_{\chi_1}}$ and $M_{\chi_2} \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times [0_{\chi_2}, u'_{\chi_2}]_{M_{\chi_2}}$. Further, we have identified elements 0_{χ_0}, 0_{χ_1} and 0_{χ_2}, and elements u'_{χ_0}, u'_{χ_1} and u'_{χ_2}. It follows that in E we have $[0_{\chi_1}, u'_{\chi_1}]_{M_{\chi_1}} \cap [0_{\chi_2}, u'_{\chi_2}]_{M_{\chi_2}} = \{0_{\chi_0}, u'_{\chi_0}\}$. We obtain that $M_{\chi_1} \cap M_{\chi_2} \cong [0_{\chi_0}, u_{\chi_0}] \times \{0_{\chi_0}, u'_{\chi_0}\} \cong F_{\chi_0}$.

(vi) By (iii), $E \cong [0, u] \times G$, where $u \in C(E)$, $[0, u]$ is a complete atomic MV-effect algebra and G is the horizontal sum of a family $(E_\chi)\chi \in H$ of complete atomic MV-effect algebras. By Riečanová (2003a), Theorem 5.2, on every complete atomic MV-effect algebra E_χ there is an (o)-continuous state ω_χ, $\chi \in H$. Let us define a mapping $\omega_G\ G \to [0, 1]$ by the following way: For every $x \in G$ let $\omega_G(x) = \omega_\chi(x)$, where $\chi \in H$ be such that $x \in E_\chi$. Obviously ω_G is an (o)-continuous state on G. Further, let ω_0 be an (o)-continuous state on $[0, u]$. For every pair of nonnegative real numbers k_1 and k_2 with $k_1 + k_2 = 1$, the mapping $\omega = k_1\omega_0 + k_2\omega_G$ is an (o)-continuous state on E. The last follows by the facts that for $x, y \in E$ with $x \le y'$ we have $(x \oplus y) \wedge u = (x \wedge u) \oplus (y \wedge u)$, as well as $(x \oplus y) \wedge u' = (x \wedge u') \oplus (y \wedge u')$ (see Riečanová, 2003a, Lemma 4.1) and $x \oplus y = ((x \oplus y) \wedge u) \oplus ((x \oplus y) \wedge u')$. Further for $x_\alpha, x \in E$ such that $x_\alpha \uparrow x$, $\alpha \in \mathcal{E}$ we have $x_\alpha \wedge u \uparrow x \wedge u$, $x_\alpha \wedge u' \uparrow x \wedge u'$ and

$x_\alpha = (x_\alpha \wedge u) \oplus (x_\alpha \wedge u') \uparrow (x \wedge u) \oplus (x \wedge u') = x$, $\alpha \in \mathcal{E}$, since there is $\chi \in H$ such that for all $\alpha \in \mathcal{E}$ we have $x_\alpha \wedge u' \in E_\chi$. □

Definition 3.6. The complete atomic effect algebra E, constructed in Theorem 3.5, is called a *pasting of MV-effect algebras* $(M_\chi)_{\chi \in \mathcal{E}}$ *together along an MV-effect algebra* $[0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \cup [u'_{\chi_0}, 1_{\chi_0}]_{M_{\chi_0}} \subseteq M_{\chi_0}$, for chosen $\chi_0 \in H$.

Remark 3.7. If in Theorem 3.5 the chosen sets D_χ of atoms of M_χ are finite then the completeness of MV-effect algebras M_χ, $\chi \in H$ can be weakened to the assumption that all M_χ are Archimedean, since then elements $u_\chi = \oplus\{n_p p \mid p \in D_\chi\}$ exist. Obviously, then $E = \cup_{\chi \in H} M_\chi$ will be an Archimedean atomic lattice effect algebra admitting an (o)-continuous state.

If M_χ, $\chi \in H$ are complete atomic Boolean algebras and all sets D_χ of atoms of M_χ have the same cardinality then $E = \cup_{\chi \in H} M_\chi$ constructed in Theorem 3.5 will be a complete atomic orthomodular lattice with blocks M_χ, $\chi \in H$. If D_χ are finite then the assumption of completeness of M_χ can be omitted. For pasting of orthomodular posets we refer the reader to Navara and Rogalewicz (1991).

4. PASTINGS OF TWO AND MORE FAMILIES OF COMPLETE ATOMIC MV-EFFECT ALGEBRAS

Assume that $(M_\chi)_{\chi \in H}$ is a family of complete atomic MV-effect algebras. Let $H_1, H_2 \subseteq H$ be nonempty sets of indices such that $H_1 \cup H_2 = H$ and $H_1 \cap H_2 = \emptyset$. Further, assume that E_1 is the pasting of $(M_\chi)_{\chi \in H_1}$ together along an MV-effect algebra $[0_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}} \cup [u'_{\chi_1}, 1_{\chi_1}]_{M_{\chi_1}}$ for chosen $\chi_1 \in H_1$, and E_2 is the pasting of $(M_\chi)_{\chi \in H_2}$ together along an MV-effect algebra $[0_{\chi_2}, u_{\chi_2}]_{M_{\chi_2}} \cup [u'_{\chi_2}, 1_{\chi_2}]_{M_{\chi_2}}$ for chosen $\chi_2 \in H_2$. By Theorem 3.5 we have $E_1 \cong [0_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}} \times G_1$ and $E_2 \cong [0_{\chi_2}, u_{\chi_2}]_{M_{\chi_2}} \times G_2$, where G_1 is the horizontal sum of all $[0_\chi, u'_\chi]_{M_\chi}$, $\chi \in H_1$ and G_2 is the horizontal sum of all $[0_\chi, u'_\chi]_{M_\chi}$, $\chi \in H_2$. Recall that here, for all $\chi \in H_1$ ($\chi \in H_2$), $u_\chi = \oplus\{n_p p \mid p \in D_\chi\} \neq 1_\chi$ and $[0_{\chi 1}, u^1_\chi]_{M_\chi} \neq \{0_{\chi 1}, u'_\chi\}$ where $D_\chi \subseteq M_\chi$ are isotropically equivalent sets of atoms.

Assume now that there are isotropically equivalent sets of atoms $K_1 \subseteq P_1 = [0_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}}$ and $K_2 \subseteq P_2 = [0_{\chi_2}, u_{\chi_2}]_{M_{\chi_2}}$. By Jenča *et al.*, 2002, Theorem 4.3, for every $p \in K_1$ ($q \in K_2$) we have $n_p p \leq u_{\chi_1}$ ($n_q q \leq u_{\chi_2}$). Moreover, by Riečanová, 2003a, Lemma 2.1, $v_1 = \oplus\{n_p p \mid p \in K_1\} = \bigvee\{n_p p \mid p \in K_1\} \leq u_{\chi_1}$ and $v_2 = \oplus\{n_q q \mid q \in K_2\} = \bigvee\{n_q q \mid q \in K_2\} \leq u_{\chi_2}$. By Theorem 3.5, (ii), $[0_{\chi_1}, v_1]_{P_1} \cong [0_{\chi_2}, v_2]_{P_2}$. Let $v_1 \neq u_{\chi_1}$ and $v_2 \neq u_{\chi_2}$. Let us identify $[0_{\chi_1}, v_1]_{P_1}$ and $[0_{\chi_2}, v_2]_{P_2}$. Further, let G_{12} be the horizontal sum of $[0_{\chi_1}, v'_1 \wedge u_{\chi_1}]_{P_1} \times G_1$ and $[0_{\chi_2}, v'_2 \wedge u_{\chi_2}]_{P_2} \times G_2$. Then there is a complete atomic effect algebra $E = \cup_{\chi \in H} M_\chi \cong [0_{\chi_1}, v_1]_{P_1} \times G_{12}$, whose family of all blocks coincides with $(M_\chi)_{\chi \in H}$. The last is a consequence of the facts that every block of $[0_{\chi_1}, v_1]_{P_1} \times G_{12}$ is a direct product of $[0_{\chi_1}, v_1]_{P_1}$ and a block of G_{12} and every block of G_{12} is isomorphic to a direct product of MV-effect algebra $[0_{\chi_1}, v_1' \wedge u_{\chi_1}]_{P_1}$ and a block of G_1, or a direct product of $[0_{\chi_2}, v_2' \wedge u_{\chi_2}]_{P_2}$ and a block of G_2 (Riečanová, 2003c). Thus every block M of E is isomorphic to $[0_{\chi_1}, v_1]_{P_1} \times [0_{\chi_1}, v_1' \wedge u_{\chi_1}]_{P_1} \times [0_\chi, u_\chi']_{M_\chi} \cong [0_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}} \times [0_\chi, u_\chi']_{M_\chi} \cong M_\chi \in (M_\chi)_{\chi \in H_1}$, for chosen $\chi_1 \in H_1$ and some $\chi \in H_1$, or $M \cong [0_{\chi_2}, v_2]_{P_2} \times [0_{\chi_2}, v_2' \wedge u_{\chi_2}]_{P_2} \times [0_\chi, u_\chi']_{M_\chi} \cong [0_{\chi_2}, u_{\chi_2}]_{M_{\chi_2}} \times [0_\chi, u_\chi']_{M_\chi} \cong M_\chi \in (M_\chi)_{\chi \in H_2}$, for chosen $\chi_2 \in H_2$ and some $\chi \in H_2$, and conversely.

Obviously, E is the pasting of E_1 and E_2 together along the MV-effect algebra $[0_{\chi_1}, v_1]_{M_{\chi_1}} \cup [v_1' \wedge u_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}}$. It follows that for all $\alpha \in H_1$ and $\beta \in H_2$ we have $M_\alpha \cap M_\beta = [0_{\chi_1}, v_1]_{M_{\chi_1}} \cup [v_1' \wedge u_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}}$, for chosen $\chi_1 \in H_1$. Further, for $\alpha, \beta \in H_1$ we have

$M_\alpha \cap M_\beta = [0_{\chi_1}, u_{\chi_1}]_{M_{\chi_1}} \cup [u_{\chi_1}', 1_{\chi_1}]_{M_{\chi_1}}$, for chosen $\chi_1 \in H_1$, and for $\alpha, \beta \in H_2$ we have

$M_\alpha \cap M_\beta = [0_{\chi_2}, u_{\chi_2}]_{M_{\chi_2}} \cup [u_{\chi_2}', 1_{\chi_2}]_{M_{\chi_2}}$, for chosen $\chi_2 \in H_2$.

The existence of an (o)-continuous state on the above constructed effect algebra E is obvious from the facts that G_{12} is a horizontal sum of two complete effect algebras admitting (o)-continuous state, hence there exists an (o)-continuous state ω_{12} on G_{12}. Further, $[0_{\chi_1}, v_1]_{P_1}$ is a complete atomic MV-effect algebra, hence there exists an (o)-continuous state m on $[0_{\chi_1}, v_1]_{P_1}$. Thus for any $0 \leq k_1, k_2 \leq 1$ with $k_1 + k_2 = 1$ the convex combination $k_1 m + k_2 \omega_{12} = \omega$ is an (o)-continuous state on E.

Obviously, the method described above can be extended for the pasting of more than two families of complete atomic MV-effect algebras.

Theorem 4.1. *Let $(M_\chi)_{\chi \in H}$ be a family of complete atomic MV-effect algebras. Let, for $\alpha \in \mathcal{E}$, $H_\alpha \subseteq H$ be such that $\cup_{\alpha \in \mathcal{E}} H_\alpha = H$ and $H_\alpha \cap H_\beta = \emptyset$, for all $\alpha \neq \beta$, $\alpha, \beta \in \mathcal{E}$. Let, for $\alpha \in \mathcal{E}$, E_α be the pasting of $(M_\chi)_{\chi \in H_\alpha}$ together along the MV-algebra $[0_{\chi_\alpha}, u_{\chi_\alpha}]_{M_{\chi_\alpha}} \cup [u_{\chi_\alpha}', 1_{\chi_\alpha}]_{M_{\chi_\alpha}}$, for chosen $\chi_\alpha \in H_\alpha$. Let $v_\alpha \in C([0_{\chi_\alpha}, u_{\chi_\alpha}]_{M_{\chi_\alpha}})$ be such that $v_\alpha \neq u_{\chi_\alpha}$ and let $[0_{\chi_\alpha}, v_\alpha]_{M_{\chi_\alpha}} \cong [0_{\chi_{\alpha_0}}, v_{\alpha_0}]_{M_{\chi_{\alpha_0}}}$, for chosen $\alpha_0 \in \mathcal{E}$ and $\chi_{\alpha_0} \in H_{\alpha_0}$, and for all $\chi_\alpha \in H_\alpha$. Let G_α be the horizontal sum of all $[0_\chi, u_\chi']_{M_\chi}$, $\chi \in H_\alpha$ and let G be the horizontal sum of effect algebras $[0_{\chi_\alpha}, v_\alpha' \wedge u_{\chi_\alpha}]_{M_{\chi_\alpha}} \times G_\alpha$, for all $\alpha \in \mathcal{E}$. Then $E = \cup_{\chi \in H} M_\chi \cong [0_{\chi_{\alpha_0}}, v_{\alpha_0}]_{M_{\chi_{\alpha_0}}} \times G$ is a complete atomic effect algebra whose family of all blocks coincides with the family $(M_\chi)_{\chi \in H}$. In this case there is an (o)-continuous state ω on E.*

The proof runs in the same manner as for the pasting of two families of complete atomic MV-effect algebras and it is left to the reader.

It is worth pointing out that the described methods of pastings of MV-effect algebras can be extended to families $(M_\chi)_{\chi \in H}$ of nonatomic MV-effect algebras with nontrivial centers and with existing $u_\chi \in C(M_\chi)$, $u_\chi \neq 1_\chi$ and $[0_{\chi 1}, u_\chi^1]_{M_\chi} \neq \{0_{\chi 1}, u_\chi'\}$ such that for chosen $\chi_0 \in H$ and all $\chi \in H$ we have $[0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \cong$

$[0_\chi, u_\chi]_{M_\chi}$. Then $E = \cup_{\chi \in H} M_\chi \cong [0_{\chi_0}, u_{\chi_0}]_{M_{\chi_0}} \times G$, where G is the horizontal sum of all $[0_\chi, u'_\chi]_{M_\chi}$, $\chi \in H$, is a lattice effect algebra whose family of all blocks coincides with $(M_\chi)_{\chi \in H}$. Since on every MV-effect algebra there is a state, the existence of a state on E can be shown.

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